

Existence & Uniqueness thm:

Given : $y' = f(x, y)$ $y(x_0) = y_0$ $R: |x - x_0| \leq a$
 $|y - y_0| \leq b$

if i) $f(x, y)$ is conti. on R

ii) $f(x, y)$ is bdd i.e. $|f(x, y)| \leq M$ $M > 0$

iii) $f(x, y)$ satisfies L.C. $\exists K > 0$

$$\text{s.t. } |f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2|$$

then $\exists$ a unique sol. $|x - x_0| \leq h$

$$h = \min\left(a, \frac{b}{M}\right)$$

eg $y' = 6x^2 y^2$ $y(0) = 1$ $R: |x| \leq 2$ $|y| \leq 3$

i) $f(x, y) = 6x^2 y^2$ is conti. bcz it is a poly.

$$\text{ii) } |f(x, y)| = |6x^2 y^2| \quad \forall |x|^n = |x|^n$$

$$\leq 6|x|^2 |y|^2 \leq 6 \cdot 2^2 \cdot 3^2 = 216 = M$$

$$\text{iii) } |f(x, y_1) - f(x, y_2)| = |6x^2 y_1^2 - 6x^2 y_2^2|$$

$$= 6|x|^2 |y_1^2 - y_2^2|$$

$$= 6|x|^2 |y_1 + y_2| |y_1 - y_2|$$

$$\leq 6 \cdot 2^2 (3+3) |y_1 - y_2|$$

$$= 144 |y_1 - y_2| \quad K = 144 > 0$$

then $\exists$ a unique sol. in $|x - 0| \leq h$

$$h = \min\left(a, \frac{b}{M}\right) = \min\left(2, \frac{3}{216}\right) = \frac{3}{216}$$

$$\boxed{|x| \leq \frac{3}{216}} \quad \underline{\text{Ans.}}$$

$$\therefore \dots \therefore y = 9/3 \quad y(0) = 1 \quad |x| \leq 2 \quad |y| \leq 3$$

eg. $y' = \frac{y^{4/3}}{3}$ $y(0) = 1$ $|x| \leq 2$ $|y| \leq 3$

$$|f(x, y)| = |y^{4/3}| = |y|^{4/3} > 3^{4/3} = M$$

$$|f(x, y_1) - f(x, y_2)| = |y_1^{4/3} - y_2^{4/3}|$$

Note: $\forall K \quad \left| \frac{\partial f}{\partial y} \right| \leq K$ then f is L.C. $K > 0$

$$\left| \frac{\partial f}{\partial y} \right| = \left| \frac{4}{3} y^{1/3} \right| = \frac{4}{3} |y|^{1/3} = \frac{4}{3} (3)^{1/3} > K > 0$$

eg. $f(x, y) = y^{2/3}$

$$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3} > \frac{2}{3} y^{1/3}$$

eg. $y' = \sin x + y^2$

$$|f(x, y)| = |\sin x + y^2| \leq |\sin x| + |y|^2 \leq 1 + 3^2 = 10 = M$$

$$|x+y| \leq |x| + |y|$$

$$|f(x, y_1) - f(x, y_2)| = |\sin x + y_1^2 - \sin x - y_2^2| = |y_1^2 - y_2^2|$$

$$\left| \frac{\partial f}{\partial y} \right| = |2y| = 2|y| \leq 2 \cdot 3 = 6 > K$$

eg. $f(x, y) = e^{2y}$ $|e^{2y}| = e^{2|y|} < \infty$

$$|f(x, y)| = |e^{2y}| \leq e^6 \quad -3 \leq y \leq 3$$

$$|\frac{\partial f}{\partial y}| = |2e^{2y}| = 2|e^{2y}| \leq 2e^6 \quad -6 \leq 2y \leq 6$$

$$\left| \frac{\partial f}{\partial y} \right| = \left| 2e^{xy} \right| > 2 \left| e^{-6} \right| \leq \underbrace{2e^6}_{< \infty} \quad \underbrace{e^{-6}}_{> 0} \leq \underbrace{e^{xy}}_{> 0} \leq e^6$$

method of variation of parameter:

$$a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = f(x)$$

$$\textcircled{1} y_c = c_1 y_1 + c_2 y_2$$

$$\left. \begin{array}{l} \textcircled{1} y_c = c_1 y_1 + c_2 y_2 \\ \textcircled{2} y_p = v_1(x) y_1 + v_2(x) y_2 \end{array} \right\} y = y_c + y_p$$

$$\textcircled{2} \text{ let } y_p = \underline{v_1(x)} y_1 + \underline{v_2(x)} y_2$$

$$\textcircled{3} v_1(x) = - \int \frac{f(x) y_2}{a_0(x) w(x)} dx \quad \textcircled{4} v_2(x) = \int \frac{f(x) y_1}{a_0(x) w(x)} dx$$

$$\textcircled{2} w(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Wronskian

Eg:

$$\frac{d^2 y}{dx^2} + 4y = \sec^2 2x$$

$$\text{C.F. } (D^2 + 4) y = 0$$

$$\text{A.E. } m^2 + 4 = 0$$

$$m = \pm 2i$$

$$\checkmark y_c = c_1 \underline{\cos 2x} + c_2 \underline{\sin 2x}$$

$$= c_1 y_1 + c_2 y_2$$

$$w(x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix}$$

$$= 2$$

$$v_1(x) = - \frac{1}{2} \int \frac{\sec^2 2x \cdot \sin 2x}{\sin^2 2x} dx$$

$$\left\{ \begin{array}{l} a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = f(x) \\ y = y_c + y_p \\ \text{Find C.F. } y_c \\ 1) \text{ Replace } \frac{d^n}{dx^n} \rightarrow D^n, \frac{d^{n-1}}{dx^{n-1}} \rightarrow D^{n-1} \dots \end{array} \right.$$

$$2) D \rightarrow m \quad \text{A.E.}$$

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n) y = 0$$

$$a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0$$

$$3) n \text{ roots } \rightarrow \lambda_1, \lambda_2, \dots, \lambda_n$$

4) i) Distinct

$$y_c = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \dots$$

$$\text{ii) Repeated } \lambda_1 = \lambda_2 = \lambda_3$$

$$2) \quad \int \frac{\sin^2 2x}{\cos^2 2x} dx$$

$$\cos 2x = t$$

$$-2 \sin 2x dx = dt$$

$$v_1(x) = \frac{1}{4} \int \frac{1}{t^2} dt$$

$$= -\frac{1}{4} \cos 2x$$

$$v_2(x) = \frac{1}{2} \int \sec^2 2x \cdot \cos 2x dx$$

$$= \frac{1}{2} \int \sec 2x dx = \frac{1}{2} \cdot \frac{1}{2} \log |\sec 2x + \tan 2x|$$

$$= \frac{1}{4} \log |\sec 2x + \tan 2x|$$

$$y_p = v_1 y_1 + v_2 y_2 = -\frac{1}{4} \cdot \cancel{\cos 2x} + \frac{1}{4} \log |\sec 2x + \tan 2x| \cdot x \sin 2x$$

$$= -\frac{1}{4} + \frac{1}{4} \sin 2x \log |\sec 2x + \tan 2x|$$

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ii) Repeated $\lambda_1 = \lambda_2 = \lambda_3$

$$y_c = (C_1 + C_2 x + C_3 x^2) e^{\lambda_1 x} + C_4 e^{\lambda_4 x} + C_5 e^{\lambda_5 x} + \dots$$

iii) complex $\alpha \pm i\beta$

$$y_c = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

iv) Surds $\alpha \pm \sqrt{\beta}$

$$y_c = e^{\alpha x} (C_1 \cosh \sqrt{\beta} x + C_2 \sinh \sqrt{\beta} x)$$